Cross Product

Goal: Given vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$, construct a new vector $\vec{w}$ which is orthogonal to both $\vec{u}$ and $\vec{v}$.

Derivation: Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be orthogonal to $\vec{w} = \langle w_1, w_2, w_3 \rangle$; we will find a solution to

$$\longrightarrow \begin{cases} \vec{u}\cdot\vec{w} = u_1w_1 + u_2w_2 + u_3w_3 = 0 \\ \vec{v}\cdot\vec{w} = v_1w_1 + v_2w_2 + v_3w_3 = 0 \end{cases}$$

We can eliminate w_3 from this equation via

$$\begin{aligned} 0 &= v_3(\vec{u}\cdot\vec{w}) - u_3(\vec{v}\cdot\vec{w}) \\ &= v_3(u_1w_1 + u_2w_2 + u_3w_3) - u_3(v_1w_1 + v_2w_2 + v_3w_3) \\ &= u_1w_1v_3 + u_2w_2v_3 - u_3v_1w_1 - u_3v_2w_2 \\ 0 &= (u_1v_3 - u_3v_1)w_1 + (u_2v_3 - u_3v_2)w_2 \end{aligned}$$

Now we observe $w_1 = u_2v_3 - v_2u_3$ and $w_2 = -(u_1v_3 - v_1u_3)$ is a solution to the above equation (form $ab - ba = 0$).

We next use our partial solution w_1, w_2 to compute w_3:

$$\begin{aligned} 0 = \vec{u}\cdot\vec{w} &= u_1w_1 + u_2w_2 + u_3w_3 \\ &= u_1(u_2v_3 - v_2u_3) - u_2(u_1v_3 - v_1u_3)w_2 + u_3w_3 \\ 0 &= u_3(-u_1v_2 + u_2v_1 + w_3) \end{aligned}$$

$\iff$ EITHER $u_3 = 0$ OR $-u_1v_2 + u_2v_1 + w_3 = 0 \iff w_3 = u_1v_2 - u_2v_1$

One easily verifies that the following is a full solution for $\vec{w}$:

$$\vec{w} = \langle u_2v_3 - v_2u_3,\ -(u_1v_3 - v_1u_3),\ u_1v_2 - u_2v_1 \rangle.$$

Def^n: The cross product of vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\vec{u} \times \vec{v} = \langle u_2 v_3 - v_2 u_3,\ -(u_1 v_3 - v_1 u_3),\ u_1 v_2 - v_1 u_2 \rangle.$$

NB: By our derivation, we have that $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

The definition of the cross product is difficult to remember... it becomes slightly easier once we translate it via matrices.

A matrix is an array of numbers.

A 2x2 matrix has determinant given by

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2$$

A 3x3 matrix has determinant given by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = +\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} a_1 - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} a_2 + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} a_3$$

If we replace the rows and do some manipulations:

$$\begin{aligned} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k} \\ &= (u_2 v_3 - v_2 u_3)\vec{i} - (u_1 v_3 - v_1 u_3)\vec{j} + (u_1 v_2 - v_1 u_2)\vec{k} \\ &= \langle u_2 v_3 - v_2 u_3,\ -(u_1 v_3 - v_1 u_3),\ u_1 v_2 - v_1 u_2 \rangle \\ &= \vec{u} \times \vec{v} \end{aligned}$$

Ex: Compute the cross product $\vec{u} \times \vec{v}$ for ...

ⓐ $\vec{u} = \langle 1,1,1 \rangle$, $\vec{v} = \langle 1,2,3 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \vec{k}$$

$$= (1 \cdot 3 - 2 \cdot 1)\vec{i} - (1 \cdot 3 - 1 \cdot 1)\vec{j} + (1 \cdot 2 - 1 \cdot 1)\vec{k}$$

$$= \langle 1, -2, 1 \rangle$$

ⓑ $\vec{u} = \langle 1, 0, 2 \rangle$, $\vec{v} = \langle \pi, 0, 2\pi \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ \pi & 0 & 2\pi \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 0 & 2\pi \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 2 \\ \pi & 2\pi \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 0 \\ \pi & 0 \end{vmatrix} \vec{k}$$

$$= (0 - 0)\vec{i} - (2\pi - 2\pi)\vec{j} + (0 - 0)\vec{k}$$

$$= \langle 0, 0, 0 \rangle = \vec{0}$$

$\vec{v} = \pi \vec{u}$

ⓒ $\vec{u} = \vec{i}$, $\vec{v} = \vec{k}$

$$\vec{i} \times \vec{k} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \vec{k}$$

$$= (0 - 0)\vec{i} - (1 - 0)\vec{j} + (0 - 0)\vec{k}$$

$$= -\vec{j}$$

Prop (Properties of Cross Product): Let $\vec{u},\vec{v},\vec{w} \in \mathbb{R}^3$ and $c \in \mathbb{R}$.

NB: $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2 = -(b_1a_2 - a_1b_2) = -\begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix}$

① $\vec{v} \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$

$$= \begin{vmatrix} v_2 & v_3 \\ u_2 & u_3 \end{vmatrix}\vec{i} - \begin{vmatrix} v_1 & v_3 \\ u_1 & u_3 \end{vmatrix}\vec{j} + \begin{vmatrix} v_1 & v_2 \\ u_1 & u_2 \end{vmatrix}\vec{k}$$

$$= -\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\vec{i} - -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\vec{j} - \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\vec{k}$$

$$= -\left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\vec{k}\right)$$

$$= -\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = -(\vec{u} \times \vec{v})$$ ▨

② $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$

NB: $\begin{vmatrix} ca_1 & ca_2 \\ b_1 & b_2 \end{vmatrix} = ca_1b_2 - b_1ca_2 = c(a_1b_2 - b_1a_2) = c\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

$(c\vec{u}) \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ cu_1 & cu_2 & cu_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

$$= \begin{vmatrix} cu_2 & cu_3 \\ v_2 & v_3 \end{vmatrix}\vec{i} - \begin{vmatrix} cu_1 & cu_3 \\ v_1 & v_3 \end{vmatrix}\vec{j} + \begin{vmatrix} cu_1 & cu_2 \\ v_1 & v_2 \end{vmatrix}\vec{k}$$

$$= c\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\vec{i} - c\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\vec{j} + c\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\vec{k}$$

$$= c\left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\vec{k}\right)$$

$$= c(\vec{u} \times \vec{v})$$

Also $\vec{u} \times (c\vec{v}) = -((c\vec{v}) \times \vec{u}) = -c(\vec{v} \times \vec{u}) = c(-(\vec{v} \times \vec{u})) = c(\vec{u} \times \vec{v})$ ▨

③ $\vec{u} \times (\vec{v} + \vec{w})$ linearity

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1+w_1 & v_2+w_2 & v_3+w_3 \end{vmatrix}$$

NB: $\begin{vmatrix} a_1 & a_2 \\ b_1+c_1 & b_2+c_2 \end{vmatrix} = a_1(b_2+c_2) - (b_1+c_1)a_2$

$= (a_1b_2 - b_1a_2) + (a_1c_2 - c_1a_2)$

$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}$

$$= \left\langle \begin{vmatrix} u_2 & u_3 \\ v_2+w_2 & v_3+w_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1+w_1 & v_3+w_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1+w_1 & v_2+w_2 \end{vmatrix} \right\rangle$$

$$= \left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} + \begin{vmatrix} u_2 & u_3 \\ w_2 & w_3 \end{vmatrix}, -\left(\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \begin{vmatrix} u_1 & u_3 \\ w_1 & w_3 \end{vmatrix} \right), \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} + \begin{vmatrix} u_1 & u_2 \\ w_1 & w_2 \end{vmatrix} \right\rangle$$

$$= \left\langle \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \right\rangle + \left\langle \begin{vmatrix} u_2 & u_3 \\ w_2 & w_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ w_1 & w_2 \end{vmatrix} \right\rangle$$

$$= (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w}) \qquad \blacksquare$$

④ $$(\vec{u} + \vec{v}) \times \vec{w} = -\left(\vec{w} \times (\vec{u} + \vec{v})\right) = -\left((\vec{w} \times \vec{u}) + (\vec{w} \times \vec{v})\right)$$

$$= -\left(-(\vec{u} \times \vec{w}) - (\vec{v} \times \vec{w})\right) = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w}) \qquad \blacksquare$$

⑤ $$\vec{u} \cdot (\vec{v} \times \vec{w}) = \langle u_1, u_2, u_3 \rangle \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= u_1(v_2w_3 - v_3w_2) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - v_2w_1)$$

$$= w_1(u_2v_3 - u_3v_2) - w_2(u_1v_3 - u_3v_1) + w_3(u_1v_2 - u_2v_1)$$

$$= \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \cdot \langle w_1, w_2, w_3 \rangle = (\vec{u} \times \vec{v}) \cdot \vec{w} \qquad \blacksquare$$

⑥ $\vec{u} \times (\vec{v} \times \vec{w}) = \langle u_1, u_2, u_3 \rangle \times \left\langle \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, -\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right\rangle$

$$= \left\langle \begin{vmatrix} u_2 & u_3 \\ -\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} & \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} & \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} & -\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \end{vmatrix} \right\rangle$$

$$= \left\langle u_2 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, -\left(u_1 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} - u_3 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \right), -u_1 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \right\rangle$$

$$= \langle u_2(v_1w_2 - w_1v_2) + u_3(v_1w_3 - w_1v_3),$$
$$-u_1(v_1w_2 - w_1v_2) + u_3(v_2w_3 - w_2v_3),$$
$$-u_1(v_1w_3 - w_1v_3) - u_2(v_2w_3 - w_2v_3) \rangle$$

$$= \langle (u_1w_1 + u_2w_2 + u_3w_3)v_1 - (u_1v_1 + u_2v_2 + u_3v_3)w_1,$$
$$(u_1w_1 + u_2w_2 + u_3w_3)v_2 - (u_1v_1 + u_2v_2 + u_3v_3)w_2,$$
$$(u_1w_1 + u_2w_2 + u_3w_3)v_3 - (u_1v_1 + u_2v_2 + u_3v_3)w_3 \rangle$$

$$= \langle (\vec{u} \cdot \vec{w})v_1 - (\vec{u} \cdot \vec{v})w_1, (\vec{u} \cdot \vec{w})v_2 - (\vec{u} \cdot \vec{v})w_2, (\vec{u} \cdot \vec{w})v_3 - (\vec{u} \cdot \vec{v})w_3 \rangle$$

$$= \langle (\vec{u} \cdot \vec{w})v_1, (\vec{u} \cdot \vec{w})v_2, (\vec{u} \cdot \vec{w})v_3 \rangle - \langle (\vec{u} \cdot \vec{v})w_1, (\vec{u} \cdot \vec{v})w_2, (\vec{u} \cdot \vec{v})w_3 \rangle$$

$$= (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$$ ////

$$\therefore \quad \vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$$

We collect the above results here for easy reference.

Prop (Properties of the Cross Product): Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and $c \in \mathbb{R}$.

① $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$

② $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$

③ $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$

④ $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$

⑤ $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$

⑥ $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

Prop (Geometric Properties of the Cross Product): Let $\vec{u}, \vec{v} \in \mathbb{R}^3$.

1. The vector $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$. ✓
2. If $\vec{u}$ and $\vec{v}$ are at angle θ, then $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin(\theta)$.
3. Vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} \times \vec{v} = \vec{0}$.

pf ②: By our previous results and definitions we have:

$$
\begin{aligned}
|\vec{u} \times \vec{v}|^2 &= (\vec{u} \times \vec{v}) \cdot (\vec{u} \times \vec{v}) \\
&= \vec{u} \cdot (\vec{v} \times (\vec{u} \times \vec{v})) \\
&= \vec{u} \cdot ((\vec{v} \cdot \vec{v}) \vec{u} - (\vec{v} \cdot \vec{u}) \vec{v}) \\
&= (\vec{v} \cdot \vec{v})(\vec{u} \cdot \vec{u}) - (\vec{v} \cdot \vec{u})(\vec{u} \cdot \vec{v}) \\
&= |\vec{v}|^2 |\vec{u}|^2 - (\vec{u} \cdot \vec{v})^2 \quad \leftarrow \quad \vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos(\theta) \\
&= (|\vec{u}||\vec{v}|)^2 - (|\vec{u}||\vec{v}| \cos(\theta))^2 \\
&= (|\vec{u}||\vec{v}|)^2 - (|\vec{u}||\vec{v}|)^2 \cos^2(\theta) \\
&= (|\vec{u}||\vec{v}|)^2 (1 - \cos^2(\theta)) \quad \leftarrow \sin^2(\theta) \\
&= (|\vec{u}||\vec{v}| \sin(\theta))^2
\end{aligned}
$$

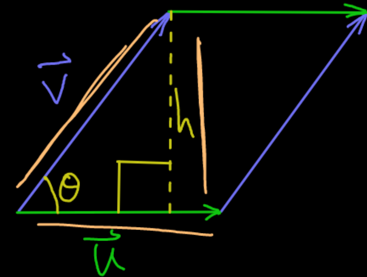
Thus $|\vec{u} \times \vec{v}|^2 = (|\vec{u}||\vec{v}| \sin(\theta))^2$; as $|\vec{u} \times \vec{v}|, |\vec{u}|, |\vec{v}| \geq 0$ and $0 \leq \theta \leq \pi$ yields $\sin(\theta) \geq 0$, we have $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin(\theta)$. //

As a result of the above theorem, we obtain several additional geometric relationships involving the cross product.

Areas of Parallelograms:

Consider the parallelogram formed by vectors $\vec{u}$ and $\vec{v}$.

The height and base of this parallelogram are $h = |\vec{v}|\sin(\theta)$ and $b = |\vec{u}|$,



yielding area $A = bh = |\vec{u}||\vec{v}|\sin(\theta) = |\vec{u}\times\vec{v}|$.

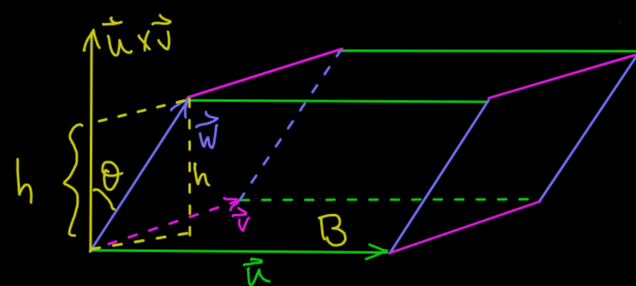
(Cor: The length of $\vec{u}\times\vec{v}$ is the area of the parallelogram formed by the vectors $\vec{u}$ and $\vec{v}$.) ☆

Volumes of Parallelopipeds:

Consider a parallelopiped determined by vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$.

We've already shown that the area of the base is $B = |\vec{u}\times\vec{v}|$.

Letting θ denote the angle



between $\vec{u}\times\vec{v}$ and $\vec{w}$, we see the height of the parallelopiped is $h = |\vec{w}||\cos(\theta)|$. Hence we compute the volume via

|scalar triple product|

$$V = Bh = |\vec{u}\times\vec{v}||\vec{w}||\cos(\theta)| = |(\vec{u}\times\vec{v})\cdot\vec{w}| = |\vec{u}\cdot(\vec{v}\times\vec{w})|.$$

NB: The quantity $\vec{u}\cdot(\vec{v}\times\vec{w})$ is the scalar triple product of the vectors $\vec{u}, \vec{v}$, and $\vec{w}$.

Ex: Compute $\vec{u} \times \vec{v}$ for the vectors...

ⓐ $\vec{u} = \langle 4, 3, -2 \rangle$, $\vec{v} = \langle 2, -1, 1 \rangle$

ⓑ $\vec{u} = \vec{i} + \vec{j}$, $\vec{v} = \vec{j} - \vec{k}$

Ex: Are the following meaningful? Why (not)?

ⓐ $\vec{u} \cdot (\vec{v} \times \vec{w})$

vector dot vector

Is meaningful!

ⓑ $\vec{u} \times (\vec{v} \cdot \vec{w})$

vector cross scalar

Not meaningful

ⓒ $\vec{u} \times (\vec{v} \times \vec{w})$

vector cross vector

Is meaningful

ⓓ $\vec{u} \cdot (\vec{v} \cdot \vec{w})$

vector dot scalar

Not meaningful

ⓔ $(\vec{u} \cdot \vec{v}) \times (\vec{u} \cdot \vec{w})$

scalar cross scalar

Not meaningful!

ⓕ $(\vec{u} \times \vec{v}) \cdot (\vec{w} \times \vec{r})$

vector dot vector

Is meaningful!

//

Ex: Find two unit vectors orthogonal to both

$$\vec{u} = \langle 3, 2, 1\rangle \quad \text{and} \quad \vec{v} = \langle -1, 1, 0\rangle.$$

Sol: leverage the cross product.

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & 1 \\ -1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix}\vec{i} - \begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix}\vec{j} + \begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix}\vec{k}$$

$$= \langle 0-1, -(0+1), 3+2\rangle = \langle -1, -1, 5\rangle$$

$$\frac{\vec{u}\times\vec{v}}{|\vec{u}\times\vec{v}|} = \frac{1}{\sqrt{1^2+1^2+5^2}}\langle -1, -1, 5\rangle = \frac{1}{3\sqrt{3}}\langle -1, -1, 5\rangle$$

$$-\frac{\vec{u}\times\vec{v}}{|\vec{u}\times\vec{v}|} = \frac{1}{3\sqrt{3}}\langle 1, 1, -5\rangle$$

← unit vectors orthogonal to both $\vec{u}$ and $\vec{v}$. ☒

Ex: Find the area of the triangle with vertices

$$P = (1, 0, 1), \quad Q = (-2, 1, 3), \text{ and } R = (4, 2, 5).$$

Sol:

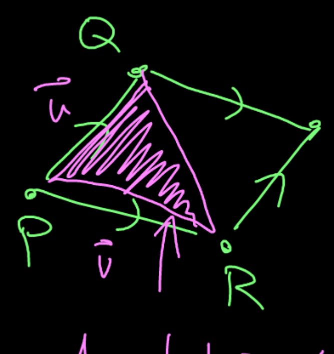
$$\vec{u} = \overrightarrow{PQ} = \langle -2-1, 1-0, 3-1\rangle = \langle -3, 1, 2\rangle$$

$$\vec{v} = \overrightarrow{PR} = \langle 4-1, 2-0, 5-1\rangle = \langle 3, 2, 4\rangle$$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}\vec{i} - \begin{vmatrix} -3 & 2 \\ 3 & 4 \end{vmatrix}\vec{j} + \begin{vmatrix} -3 & 1 \\ 3 & 2 \end{vmatrix}\vec{k}$$

$$= \langle 0, 18, -9\rangle$$

Picture



$$A = \frac{1}{2}|\vec{u}\times\vec{v}|$$

$$\therefore A = \frac{1}{2}|\vec{u}\times\vec{v}| = \frac{1}{2}|9\langle 0, 2, -1\rangle| = \frac{9}{2}\sqrt{2^2+(-1)^2} = \frac{9}{2}\sqrt{5}$$ ☒

Ex: Compute the volume of the parallelopiped determined by the vectors $\vec{u} = \langle 1,2,3 \rangle$, $\vec{v} = \langle -1,1,2 \rangle$, $\vec{w} = \langle 2,1,4 \rangle$.

Sol: The volume of the corresponding parallelopiped is

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})| = \text{abs} \begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix}$$

$$= \text{abs}\left(\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} \cdot 1 - \begin{vmatrix} -1 & 2 \\ 2 & 4 \end{vmatrix} \cdot 2 + \begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} \cdot 3 \right)$$

$$= |(4-2) - (-4-4) \cdot 2 + (-1-2) \cdot 3|$$

$$= |2 + 16 - 9| = |9| = 9$$

$\blacksquare$

Ex: Find all vectors $\vec{v}$ such that $\langle 1,2,1 \rangle \times \vec{v} = \langle 3,1,-5 \rangle$.

Sol: $\langle 3,1,-5 \rangle = \langle 1,2,1 \rangle \times \langle x,y,z \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ x & y & z \end{vmatrix}$

$$= \langle 2z - y, -(z-x), y - 2x \rangle$$

$$\therefore \begin{cases} 2z - y = 3 \\ x - z = 1 \\ y - 2x = -5 \end{cases} \iff \begin{cases} x = 1 + z \\ 2z - y = 3 \\ y - 2(1+z) = -5 \end{cases} \iff \begin{cases} x = 1 + z \\ y = 2z - 3 \\ (2z-3) - 2(1+z) = -5 \end{cases}$$

$$\iff \begin{cases} x = 1 - z \\ y = 2z - 3 \\ -5 = -5 \end{cases} \iff \begin{cases} x = 1 - t \\ y = 2t - 3 \\ z = t \end{cases}$$

$\therefore \vec{v} = \langle 1-t, 2t-3, t \rangle$ for any $t \in \mathbb{R}$. $\blacksquare$

($-5 = -5$ ← true! no constraint on z!)

Ex: Why aren't there any vectors $\vec{v}$ with $\langle 1,2,1 \rangle \times \vec{v} = \langle 3,1,5 \rangle$?

Sol: If $\langle 1,2,1 \rangle \times \vec{v} = \langle 3,1,5 \rangle$, then $\langle 1,2,1 \rangle \cdot \langle 3,1,5 \rangle = 0$

But $\langle 1,2,1 \rangle \cdot \langle 3,1,5 \rangle = 3 + 2 + 5 = 10 \neq 0$.

Hence no such $\vec{v}$ exists! $\blacksquare$

Ex: Suppose $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and $\vec{u} \neq \vec{0}$.

(a) If $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$, can we conclude $\vec{v} = \vec{w}$?

(b) If $\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$, can we conclude $\vec{v} = \vec{w}$?

(c) If $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$ AND $\vec{u} \times \vec{v} = \vec{u} \times \vec{w}$, can we conclude $\vec{v} = \vec{w}$?